

# ON LOCALLY SYMMETRIC 3-DIMENSIONAL RIEMANNIAN LIE GROUPS.

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**ABSTRACT.** In this paper, we use the powerful tool Milnor bases to classify all the 3-dimensional connected and locally symmetric Riemannian Lie Groups by solving system of polynomial equations of structure constants of each Lie algebra. Moreover, we showed that  $E_0(2)$ , is the only Lie group with locally symmetric left invariant Riemannian metrics which are not symmetric.

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## 1. INTRODUCTION AND MAIN RESULTS

A Lie group  $G$  together with Left invariant Riemannian metric  $g$  is called a *Riemannian Lie group*. Let  $\nabla, R$  and  $\langle, \rangle$  denoted the Levi-Civita connexion, the Riemann curvature tensor associated to  $g$  and the inner product induced by  $g$  on the Lie algebra  $\mathfrak{g}$  of  $G$ , respectively. The left invariant Riemannian metric  $g$  on  $G$  defines an inner product on the Lie algebra  $\mathfrak{g}$  of  $G$ , and conversely, any inner product on  $\mathfrak{g}$  gives rise to a unique left invariant metric on  $G$ . In [5], Milnor give a complete classification of 3-dimensional metric Lie algebras. Ku Yong Ha and Jong Bum Lee in [3] classified up to automorphism all left invariant Riemannian metrics on a 3-dimensional simply connected Lie groups.

A locally symmetric Riemannian Lie group is defined as a Riemannian Lie group for which the geodesic symmetric is a local isometry. This is equivalent to saying that:

**Definition 1.1.** [6] *A left invariant Riemannian metric  $g$  on a Lie group  $G$  is locally symmetric if  $\nabla R = 0$ .*

The relation  $\nabla R = 0$  means precisely that for  $x, y, z, w \in \mathfrak{g}$ ,

$$\nabla_w(R(x, y)z) = R(\nabla_w x, y)z + R(x, \nabla_w y)z + R(x, y)\nabla_w z, \quad (1)$$

where  $R(x, y) = \nabla_{[x, y]} - \nabla_x \nabla_y + \nabla_y \nabla_x$ .

We find what conditions on the real entry of the matrix (up to automorphism) of the inner product  $\langle, \rangle$  are needed to the associated metric  $g$  to be locally symmetric. For this purpose, we use the equation (1) of Definition 1.1 and the classification of Ha and Lee in [3]. Now we now formulate the following results.

**Theorem 1.2.** *We have the following:*

- (1) *All the left invariant Riemannian metrics on the Lie groups  $\mathbb{R}^3$  or  $G_I$  are locally symmetric.*

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- (2) The locally symmetric left invariant Riemannian metrics on  $\widetilde{E}_0(2)$  are equivalent up to automorphism to the metric who associated matrix is of the form  $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \nu \end{pmatrix}$ ,  $\nu > 0$ .
- (3) The locally symmetric left invariant Riemannian metrics on  $SU(2)$  are equivalent up to automorphism to the metric who associated matrix is of the form  $\lambda I_3$ ,  $\lambda > 0$ .
- (4) The locally symmetric left invariant Riemannian metrics on  $G_D$  are equivalent up to automorphism to the metric who associated matrix is of the form:
- (a)  $\begin{pmatrix} 1 & \frac{1}{2} & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & \nu \end{pmatrix}$ ,  $\nu > 0$ , if  $D = 0$ ;
- (b)  $\begin{pmatrix} 1 & 1 & 0 \\ 1 & D & 0 \\ 0 & 0 & \nu \end{pmatrix}$ ,  $\nu > 0$ , if  $D > 1$ .

We obtain the following result:

**Theorem 1.3.** Let  $g$  be a left invariant Riemannian metric on  $E_0(2)$ ,

- (1)  $(E_0(2), g)$  is a symmetric Riemannian Lie group if and only if there exist on the Lie algebra of the Lie group  $E_0(2)$ , a basis in which the associated matrix of the metric  $g$  is  $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \nu \end{pmatrix}$  where  $\frac{1}{\sqrt{\nu}} \in \mathbb{N} \setminus \{0\}$ .
- (2)  $(E_0(2), g)$  is a locally symmetric Riemannian Lie group and non symmetric space if and only if there exist on the Lie algebra of the Lie group  $E_0(2)$ , a basis in which the associated matrix of the metric  $g$  is  $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \nu \end{pmatrix}$ ,  $\nu > 0$ ,  $\frac{1}{\sqrt{\nu}} \notin \mathbb{N} \setminus \{0\}$ .

This paper is organized as follow, in section 2, we give the preliminaries, in section 3 we give the proof of Theorem 1.2 and section 4 is devoted to the proof of Theorem 1.3.

## 2. PRELIMINARIES

In this section, we recall all the necessary tools to establish proof of theorems 1.2 and 1.3.

**2.1. The Lie groups  $G_I$  and  $G_D$ .** The two non isomorphic 3-dimensional non unimodular Lie algebras  $\mathfrak{g}_I$  and  $\mathfrak{g}_D$ , given in [5], and [3] are described in the following table, with Lie brackets in the canonical basis  $(X_1, X_2, X_3)$ . The table also indicated the associated simply connected Lie group. The map  $\varphi_D$  in this table is defined by:

$$\varphi_D(t) = e^{t \frac{e^{zt} + e^{-zt}}{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + e^{t \frac{e^{zt} - e^{-zt}}{2z}} \begin{pmatrix} -1 & -D \\ 1 & 1 \end{pmatrix} \text{ with } z = \sqrt{1-D} \neq 0 \text{ and}$$

$$\varphi_1(t) = e^t \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + e^t t \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}$$

$N^\circ$	Lie algebra and Lie bracket	simply connected associated lie group
1	<p>nonunimodular solvable <math>\mathfrak{g}_I \cong \mathbb{R}^2 \rtimes_{\sigma_I} \mathbb{R}</math></p> $\sigma_I(t) = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}$ <p><math>[X_1, X_2] = 0, [X_3, X_1] = X_1, [X_3, X_2] = X_2</math></p>	$G_I \cong \mathbb{R}^2 \rtimes_{\varphi_I} \mathbb{R}$ $\varphi_I(t) = \begin{pmatrix} e^t & 0 \\ 0 & e^t \end{pmatrix}$
2	<p>nonunimodular solvable <math>\mathfrak{g}_D \cong \mathbb{R}^2 \rtimes_{\sigma_D} \mathbb{R}</math></p> $\sigma_D(t) = \begin{pmatrix} 0 & -Dt \\ t & 2t \end{pmatrix}$ <p><math>[X_1, X_2] = 0, [X_3, X_1] = X_2,</math>  <math>[X_3, X_2] = -DX_1 + 2X_2, D \in \mathbb{R}</math></p>	$G_D \cong \mathbb{R}^2 \rtimes_{\varphi_D} \mathbb{R}$

Table 1

For more details about description of 3–dimensional unimodular and non unimodular Lie groups, see [5, 3].

**2.2. Milnor bases.** We recall the classification 3–dimensional real metric Lie algebras.

**Lemma 2.1.** [5] *Let  $G$  be a connected unimodular 3–dimensional Riemannian Lie group and  $\mathfrak{g}$  his Lie algebra. There exist on  $\mathfrak{g}$ , an  $\langle \cdot, \cdot \rangle$ –orthonormal basis  $(e_1, e_2, e_3)$  such that the Lie braket are defined by:*

$$[e_1, e_2] = a e_3, \quad [e_2, e_3] = c e_1, \quad [e_3, e_1] = b e_2;$$

**Remark 2.2.** *For these Lie algebras, Milnor prove the existence of a basis in which at most one of the structure constants  $a, b, c \in \mathbb{R}$  is negative.*

**Lemma 2.3.** [5] *Let  $G$  be a connected non-unimodular 3–dimensional Riemannian Lie group and  $\mathfrak{g}$  his Lie algebra. There exist on  $\mathfrak{g}$ , an  $\langle \cdot, \cdot \rangle$ –orthonormal basis  $(e_1, e_2, e_3)$  such that the Lie braket is defined by:*

$$[e_1, e_2] = a e_2 + b e_3, \quad [e_1, e_3] = c e_2 + d e_3, \quad [e_2, e_3] = 0 \quad \text{with} \quad a + d \neq 0 \quad \text{and} \quad ac + bd = 0.$$

**Remark 2.4.** (1) *The structure constants  $a, b, c$  and  $d$  are uniquely determined, if we normalized by requiring that  $a \geq d, \quad b \geq c, \quad \text{and} \quad a + d > 0$ .*

(2) *For the Lie algebra  $\mathfrak{g}_D$ , the complete isomorphism invariant (the determinant  $D$  of  $\text{ad}_{X_3}$ ) is given by:*

$$D = \frac{4(ad - bc)}{(a + d)^2}. \quad (2)$$

**Definition 2.5.** *The bases given by Lemma 2.1 and Lemma 2.3 are called Milnor bases.*

**2.3. Locally symmetric 3-dimensional Riemannian Lie groups.** We give an algebraic characterization of 3–dimensional locally symmetric Riemannian Lie groups from the structure constants of the associated Lie algebras, respectively.

**Proposition 2.6.** *Let  $G$  be a connected 3–dimensional real unimodular Lie group with left-invariant Riemannian metric.  $(G, g)$  is a locally symmetric Riemannian Lie group if and only if in the Lie algebra  $\mathfrak{g}$  of  $G$ , there exists an  $\langle \cdot, \cdot \rangle$ –orthonormal basis in which structure constants of the Lie algebra are presented in the following table:*

Lie algebra	structure constants	restrictions
$\mathbb{R}^3$	$C_{i,j}^k = 0$	
$\mathbb{R}^2 \rtimes \mathfrak{so}(2)$	$C_{1,2}^3 = C_{3,1}^2 = a,$	$a > 0$
$\mathfrak{su}(2)$	$C_{1,2}^3 = C_{2,3}^1 = C_{3,1}^2 = a$	$a > 0$

Table 2

**proof:** Using orthonormal Milnor basis, see [5] for unimodular euclidian Lie algebras such that the structure constants are  $C_{1,2}^3 = a$ ,  $C_{2,3}^1 = c$ ,  $C_{3,1}^2 = b$ , the non-null components of the Riemannian curvature tensor  $R$  are:

$$\begin{aligned} R(e_1, e_2)e_1 &= \frac{-2a(a-b-c) - (a-b+c)(a+b-c)}{4} e_2; \\ R(e_1, e_2)e_2 &= \frac{2a(a-b-c) + (a-b+c)(a+b-c)}{4} e_1; \\ R(e_1, e_3)e_1 &= \frac{2b(a-b+c) + (a-b-c)(a+b-c)}{4} e_3; \\ R(e_1, e_3)e_3 &= \frac{-2b(a-b+c) - (a-b-c)(a+b-c)}{4} e_1; \\ R(e_2, e_3)e_2 &= \frac{2c(a+b-c) + (a-b+c)(a-b-c)}{4} e_3; \\ R(e_2, e_3)e_3 &= \frac{-2c(a+b-c) - (a-b+c)(a-b-c)}{4} e_2. \end{aligned}$$

For the vanishing component  $R(e_1, e_2)e_3$ ,  $R(e_1, e_3)e_2$ ,  $R(e_2, e_3)e_1$ , if the metric is locally symmetric, then the direct computation of the local symmetry condition, see equation (1) in Definition 1.1, yields the system:

$$\begin{aligned} (a-b)(a+b-c)^2 &= 0 \\ (c-a)(a-b+c)^2 &= 0 \\ (c-b)(a-b-c)^2 &= 0 \end{aligned} \quad (3)$$

By Remark 2.2,  $(a, b, c)$  is a solution of (3) if and only if  $(a, b, c) \in \{(0, b, b), (a, a, 0), (a, 0, a), (a, a, a), a, b \in \mathbb{R}^{>0}\}$ .

- (1) If  $(a, b, c) \in \{(0, b, b), (a, a, 0), (a, 0, a); a, b \in \mathbb{R}\}$ , then the curvature tensor vanish i.e  $R(u, v)w = 0$  for all  $u, v, w \in \mathfrak{g}$ . Thus,  $\nabla R = 0$  and the metric is locally symmetric.
- (2) If  $(a, b, c) = (a, a, a)$  with  $a \neq 0$ , then the non vanishing components of curvature tensor are:

$$\begin{aligned} R(e_1, e_2)e_1 &= \frac{1}{2}a^2e_2, & R(e_1, e_3)e_2 &= \frac{1}{2}a^2e_3, & R(e_2, e_3)e_2 &= \frac{1}{2}a^2e_3, \\ R(e_1, e_2)e_2 &= -\frac{1}{2}a^2e_1, & R(e_1, e_3)e_3 &= -\frac{1}{2}a^2e_1, & R(e_2, e_3)e_1 &= -\frac{1}{2}a^2e_2. \end{aligned}$$

By direct computation, the equality

$$\nabla_{e_m}(R(e_i, e_j)e_k) = R(e_i, e_j)\nabla_{e_m}e_k + R(\nabla_{e_m}e_i, e_j)e_k + R(e_i, \nabla_{e_m}e_j)e_k$$

holds for  $i, j, k, m \in \{1, 2, 3\}$ .

Therefore the metric is locally symmetric. ■

**Proposition 2.7.** *Let  $G$  be a connected 3-dimensional real nonunimodular Lie group with left-invariant Riemannian metric.  $(G, g)$  is a locally symmetric Riemannian Lie group if and only if in the Lie algebra  $\mathfrak{g}$  of  $G$ , there exist an  $\langle, \rangle$ -orthonormal basis in which the structure constants of the Lie algebra are presented in the following table:*

Lie algebra	structure constants	restrictions
$\mathbb{R}^4$	commutative algebra: $C_{i,j}^k = 0$	
$\mathfrak{g}_I$	$C_{1,2}^2 = C_{1,3}^3 = a$	$a > 0$
$\mathfrak{g}_D$	$C_{1,2}^2 = C_{1,3}^3 = a, \quad C_{1,2}^3 = -C_{1,3}^2 = b \quad \text{or} \quad C_{1,3}^2 = a$	$a > 0, b > 0$

Table 3

**proof:** Using an orthonormal Milnor basis for nonunimodular euclidian Lie algebras such that the structure constants are  $C_{1,2}^2 = a, \quad C_{1,2}^3 = b, \quad C_{1,3}^2 = c, \quad C_{1,3}^3 = d$  with  $a + d \neq 0$  and  $ac + bd = 0$ . The non null components of the curvature tensor  $R$  are:

$$\begin{aligned}
 R(e_1, e_2)e_1 &= -(a^2 + \frac{3}{4}b^2 - \frac{1}{4}c^2 + \frac{1}{2}bc)e_2, & R(e_1, e_2)e_2 &= (a^2 + \frac{3}{4}b^2 - \frac{1}{4}c^2 + \frac{1}{2}bc)e_1, \\
 R(e_1, e_3)e_1 &= -(d^2 - \frac{1}{4}b^2 + \frac{3}{4}c^2 + \frac{1}{2}bc)e_3, & R(e_1, e_3)e_3 &= (d^2 - \frac{1}{4}b^2 + \frac{3}{4}c^2 + \frac{1}{2}bc)e_1, \\
 R(e_2, e_3)e_2 &= (\frac{1}{4}(b+c)^2 - ad)e_3, & R(e_2, e_3)e_3 &= -(\frac{1}{4}(b+c)^2 - ad)e_2.
 \end{aligned}$$

if the metric is locally symmetric, then the direct computation of the local symmetry condition of equation (1) in Definition 1.1, yields the system:

$$\begin{aligned}
 (b-c)(a^2 + b^2 - c^2 - d^2) &= 0 \\
 (b+c)(a^2 + b^2 - ad + bc) &= 0 \\
 d(a^2 + b^2 - ad + bc)^2 &= 0 \\
 a(c^2 + d^2 - ad + cb) &= 0 \\
 (b+c)(c^2 + d^2 - ad + bc) &= 0 \\
 ac + bd &= 0 \\
 a + d &\neq 0
 \end{aligned} \tag{4}$$

Using computer system Maple, the set of non trivial and real solutions of the system (4) is

$\{(a, b, -b, a), (0, 0, 0, d), (a, 0, 0, 0); a, d \in \mathbb{R}^*, b \in \mathbb{R}\}$ . By Remark 2.4, the set of non trivial solution of system (4) is  $\{(a, b, -b, a), (a, 0, 0, 0), a > 0, b > 0\}$  ■

**Remark 2.8.** (1) From the description of unimodular Lie groups in [5], the simply connected real unimodular Lie groups that are suppose to admit locally symmetric left invariant Riemannian metrics are, either  $\mathbb{R}^3$  or the group  $\tilde{E}_0(2)$  or the group  $SU(2)$ ,

(2) If  $(a, b, c, d) = (a, 0, 0, a), a \in \mathbb{R}^{>0}$ , then the Lie algebra  $\mathfrak{g}$  is isomorphic  $\mathfrak{g}_I$ . Otherwise,

(3) the complete isomorphism invariant  $D$  is given by:

$$\begin{cases} D = 1 + \left(\frac{b}{a}\right)^2 > 1 & \text{if } (a, b, c, d) = (a, b, -b, a), \quad a, b \in \mathbb{R}^{>0} \quad \text{or} \\ D = 0 & \text{if } (a, b, c, d) = (a, 0, 0, 0), \quad a \in \mathbb{R}^{>0}. \end{cases}$$

(4) If  $D \leq 1$  and  $D \neq 0$ , then  $(G_D, g)$  is not a locally symmetric Riemannian Lie group.

The above remark will be very usefull for the investigation of locally symmetric 3-dimensional Riemannian Lie groups.

**2.4. Riemannian symmetric spaces.** Let  $(M, g)$  be a Riemannian manifold geodesically complete. From [9], we have:

**Definition 2.9.** Let  $M$  be a Riemannian manifold, and let  $x \in M$ . Fix a star-shaped, symmetric neighborhood  $V$  of 0 in  $T_x M$ , such that the exponential map  $\exp_x$  maps  $V$  diffeomorphically

onto a neighborhood  $U$  of  $x$  in  $M$ . The local geodesic symmetry at  $x$  is the diffeomorphism  $S_x$  of  $U$  defined by:

$$S_x(y) = \exp_x \circ (-id) \circ (\exp_x)^{-1}(y) \quad (5)$$

for all  $y \in U$ .

For a symmetric space we have from [6]:

**Definition 2.10.** A Riemannian symmetric space is a connected Riemannian manifold  $M$  such that for each  $x \in M$ , there exist a (unique) isometry  $\tau_x : M \rightarrow M$  with differential map  $-Id$  on  $T_x M$ .

**Remark 2.11.** Since the differential map of  $\tau_x$  on  $T_x M$  is  $-Id$ , we obtain that  $\tau_x(x) = x$ . Let us suppose now, that the locally geodesic symmetric  $S_x$  at  $x$  is a local isometry of  $M$ . Since any isometry is determined by its value and its derivative at a single point,  $S_x = \tau_x$ . Therefore  $\tau_x$  is the unique extension of  $S_x$  on  $M$ .

For the particular case of a Riemannian Lie group  $(G, g)$ , we have:

**Remark 2.12.** Let  $e$  be the identity element of  $G$ . We have:

- (1)  $L_x \circ \exp_e = \exp_{L_x(e)} \circ L_{x*e}$  (see [6]) and,  $\exp_x = L_x \circ \exp_e \circ (L_{x*e})^{-1}$ .
- (2) For all  $x \in G$ ,  $S_x = L_x \circ S_e \circ L_x^{-1}$  where  $S_e$  is the local geodesic symmetry at  $e$ .
- (3) If  $g$  is a bi-invariant Riemannian metric, the symmetry at the unit element  $e$  is the inversion i.e.  $S_e(x) = x^{-1}$  and  $(G, g)$  is a symmetric space.

For more details on symmetric spaces see [6, 7, 8]

**2.5. Geodesics on Riemannian Lie Groups.** Let  $G$  be a Lie group,  $\mathfrak{g}$  its Lie algebra and  $\gamma : t \mapsto \gamma(t)$  a  $C^\infty$  curve of  $G$ ,  $\dot{\gamma}(t) \in T_{\gamma(t)}G$  and  $(L_{\gamma(t)*e})^{-1}\dot{\gamma}(t)$  is an element of the Lie algebra  $\mathfrak{g}$ . Setting  $\alpha(t) = (L_{\gamma(t)*e})^{-1}\dot{\gamma}(t)$ , we have the following equation:

$$\dot{\gamma}(t) = L_{\gamma(t)*e}\alpha(t). \quad (6)$$

Therefore any  $C^\infty$  curve  $\gamma$  on  $G$  induces a  $C^\infty$  curve  $\alpha$  on  $\mathfrak{g}$  that satisfies the equation (6). Let  $g$  be a left invariant Riemannian metric on  $G$ .

**Proposition 2.13.** [1, 2]  $\gamma$  is a geodesic if and only

$$\dot{\alpha}(t) = (ad_{\alpha(t)})^*\alpha(t) \quad (7)$$

where  $(ad_{\alpha(t)})^*$  is the adjoint of the linear operator  $ad_{\alpha(t)}$  with respect to inner product  $\langle, \rangle$  on  $\mathfrak{g}$ .

If  $(e_1, e_2, \dots, e_n)$  is an orthonormal basis of  $\mathfrak{g}$  with respect to the inner product  $\langle, \rangle$ , the equation (7) is equivalent to:

$$\dot{\alpha}(t) = \sum_{k=1}^{k=n} \langle \alpha(t), [ \alpha(t), e_k ] \rangle e_k \quad (8)$$

### 3. CLASSIFICATION OF LOCALLY SYMMETRIC LEFT INVARIANT RIEMANNIAN METRICS

In this section we classify all the locally symmetric left invariant metrics on the 3-dimensional simply connected Lie groups. This classification is a proof for Theorem 1.2. For this purpose we use the classification of left invariant Riemannian metrics up to automorphism given by Ha and Lee, see [3]. Let  $(g_{ij})$  be the matrix of the inner product  $\langle, \rangle$  induced by the metric  $g$  on  $\mathfrak{g}$  with respect to the canonical basis  $(X_1, X_2, X_3)$ , see



[3] ; If  $(e_1, e_2, e_3)$  is the orthonormal Milnor basis, see [5], and  $(g'_{ij})$  the matrix of  $g_e$  with respect to  $(e_1, e_2, e_3)$ , then:

$$(g'_{ij}) = P^t(g_{ij})P \quad (9)$$

where

$$(e_1, e_2, e_3) = (X_1, X_2, X_3)P, \text{ with } P = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad a_{ij} \in \mathbb{R}. \quad (10)$$

**3.1. Unimodular 3-dimensional Riemannian Lie groups.** The Lie algebra  $\mathfrak{g}$  of non trivial unimodular Lie group has basis  $(X_1, X_2, X_3)$  such that:

$$[X_1, X_2] = 0 \quad [X_3, X_1] = -X_2 \quad [X_3, X_2] = X_1$$

in this case,  $\mathfrak{g} = \mathbb{R} \rtimes \mathfrak{so}(2)$  and the simply connected associated Lie group is  $\widetilde{E}_0(2)$  or

$$[X_1, X_2] = X_3 \quad [X_3, X_1] = X_2 \quad [X_3, X_2] = -X_1$$

in this case,  $\mathfrak{g} = \mathfrak{so}(3)$  or  $\mathfrak{su}(2)$  and the simply connected associated Lie group is  $SU(2)$  (see [3]). In the rest of this subsection,  $(e_1, e_2, e_3)$  is a Milnor basis for 3-dimensional unimodular Lie algebra.

3.1.1. *Lie group  $\widetilde{E}_0(2)$ .* Any left invariant metric on  $\widetilde{E}_0(2)$  is equivalent up to automorphism to a metric whose associated matrix is of the form  $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{pmatrix}$ ,  $0 < \mu \leq 1$ ,  $\nu > 0$ .

(see [3])

A left invariant metric  $g$  on  $\widetilde{E}_0(2)$  is locally symmetric if there exist a Milnor basis relative to  $\langle, \rangle$  which satisfied (9) and (10) with constant structures of the form  $(a, b, c) = (a, a, 0)$ , with  $a > 0$  by Proposition 2.6. The below polynomial system then follow:

$$\begin{aligned} aa_{13} - (a_{31}a_{22} - a_{21}a_{32}) &= 0 \\ aa_{23} + a_{31}a_{12} - a_{11}a_{32} &= 0 \\ aa_{33} &= 0 \\ a_{32}a_{23} - a_{22}a_{33} &= 0 \\ a_{32}a_{13} - a_{12}a_{33} &= 0 \\ aa_{12} - (a_{33}a_{21} - a_{23}a_{31}) &= 0 \\ aa_{22} + a_{33}a_{11} - a_{31}a_{13} &= 0 \\ aa_{32} &= 0 \\ a_{11}^2 + a_{21}^2\mu + a_{31}^2\nu - 1 &= 0 \\ a_{11}a_{12} + a_{21}\mu a_{22} + a_{31}\nu a_{32} &= 0 \\ a_{11}a_{13} + a_{21}\mu a_{23} + a_{31}\nu a_{33} &= 0 \\ a_{12}^2 + a_{22}^2\mu + a_{32}^2\nu - 1 &= 0 \\ a_{12}a_{13} + a_{22}\mu a_{23} + a_{32}\nu a_{33} &= 0 \\ a_{13}^2 + a_{23}^2\mu + a_{33}^2\nu - 1 &= 0 \end{aligned} \quad (11)$$

This system of polynomials equations hold if and only  $\mu = 1$  and we can choose

$$P = \begin{pmatrix} 0 & -a_{23} & a_{13} \\ 0 & a_{13} & a_{23} \\ \frac{1}{\sqrt{\nu}} & 0 & 0 \end{pmatrix}, \text{ with } a_{23}, a_{13} \in \mathbb{R} \text{ and } a_{23}^2 + a_{13}^2 = 1.$$

The Riemannian Lie group  $(\widetilde{E}_0(2), g)$  is locally symmetric if and only if the metric  $g$  is equivalent up to automorphism to the metric who associated matrix is  $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \nu \end{pmatrix}, \nu > 0$ .

3.1.2. *Lie group  $SU(2)$ .* Any left invariant metric on  $SU(2)$  is equivalent up to automorphism to a metric whose associated matrix is of the form  $\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{pmatrix}$  where  $\lambda \geq \mu \geq \nu > 0$ . (see [3])

A left invariant metric  $g$  on  $SU(2)$  is locally symmetric if there exist a Milnor basis relative to  $\langle , \rangle$  which satisfied (9) and (10) with constant structures of the form  $(a, b, c) = (a, a, a)$ , with  $a > 0$  by Proposition 2.6. These relations yield the following system:

$$\begin{aligned} a a_{13} + a_{31} a_{22} - a_{21} a_{32} &= 0 \\ a a_{23} - (a_{31} a_{12} - a_{11} a_{32}) &= 0 \\ a a_{33} - (a_{11} a_{22} - a_{21} a_{12}) &= 0 \\ a a_{11} + a_{32} a_{23} - a_{22} a_{33} &= 0 \\ a a_{21} - (a_{32} a_{13} - a_{12} a_{33}) &= 0 \\ a a_{31} - (a_{12} a_{23} - a_{22} a_{13}) &= 0 \\ a a_{12} + a_{33} a_{21} - a_{23} a_{31} &= 0 \\ a a_{22} - (a_{33} a_{11} - a_{13} a_{31}) &= 0 \\ a a_{32} - (a_{13} a_{21} - a_{23} a_{11}) &= 0 \end{aligned} \tag{12}$$

$$\begin{aligned} a_{11}^2 \lambda + a_{21}^2 \mu + a_{31}^2 \nu - 1 &= 0 \\ a_{11} \lambda a_{12} + a_{21} \mu a_{22} + a_{31} \nu a_{32} &= 0 \\ a_{11} \lambda a_{13} + a_{21} \mu a_{23} + a_{31} \nu a_{33} &= 0 \\ a_{12}^2 \lambda + a_{22}^2 \mu + a_{32}^2 \nu - 1 &= 0 \\ a_{12} \lambda a_{13} + a_{22} \mu a_{23} + a_{32} \nu a_{33} &= 0 \\ a_{13}^2 \lambda + a_{23}^2 \mu + a_{33}^2 \nu - 1 &= 0 \end{aligned}$$

This system of polynomials equations hold if and only  $\lambda = \mu = \nu$ ,  $\lambda > 0$  and we can choose

$$P = \begin{pmatrix} \frac{1}{\sqrt{\lambda}} & 0 & 0 \\ 0 & \frac{-1}{\sqrt{\lambda}} & 0 \\ 0 & 0 & \frac{-1}{\sqrt{\lambda}} \end{pmatrix}, \text{ with } \lambda \in \mathbb{R}^{>0}.$$

The Riemannian Lie group  $(SU(2), g)$  is locally symmetric if and only if the metric  $g$  is equivalent up to automorphism to the metric who associated matrix is  $\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \lambda > 0$ .

**Remark 3.1.** *This metrics are also bi-invariant.*

**3.2. Non-unimodular 3-dimensional Riemannian Lie groups.** In the rest of this subsection,  $(e_1, e_2, e_3)$  is a Milnor basis for 3-dimensional non-unimodular Lie algebra.



3.2.1. *Riemannian Lie group  $G_I$ .* Any left invariant metric on  $G_I$  is equivalent up to automorphism to a metric whose associated matrix is of the form  $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \nu \end{pmatrix}$   $\nu > 0$ .

(see [3])

A left invariant metric  $g$  on  $G_I$  is locally symmetric if there exist a Milnor basis relative to  $\langle , \rangle$  which satisfied (9) and (10) with constant structures of the form  $(a, b, c, d) = (a, 0, 0, a)$ , with  $a > 0$  by Proposition 2.7. Moreover, we have the below polynomial system:

$$\begin{aligned}
 a a_{12} - (a_{31}a_{12} - a_{11}a_{32}) &= 0 \\
 a a_{22} - (a_{31}a_{22} - a_{21}a_{32}) &= 0 \\
 a a_{32} &= 0 \\
 a a_{13} - (a_{31}a_{12} - a_{11}a_{32}) &= 0 \\
 a a_{23} - (a_{31}a_{23} - a_{21}a_{33}) &= 0 \\
 a a_{33} &= 0 \\
 a_{11}^2 + a_{21}^2 + a_{31}^2\nu - 1 &= 0 \\
 a_{11}a_{12} + a_{22}a_{21} + a_{31}\nu a_{32} &= 0 \\
 a_{11}a_{13} + a_{21}a_{23} + a_{31}\nu a_{33} &= 0 \\
 a_{12}^2 + a_{22}^2 + a_{32}^2\nu &= 0 \\
 a_{12}a_{13} + a_{22}a_{23} + a_{32}\nu a_{33} &= 0 \\
 a_{13}^2 + a_{23}^2 + a_{33}^2\nu &= 0
 \end{aligned} \tag{13}$$

This system of polynomials equations hold for all  $\nu > 0$  and we can choose  $P = \begin{pmatrix} 0 & 1 & 1 \\ 0 & a_{22} & 0 \\ \frac{1}{\sqrt{\nu}} & 0 & 0 \end{pmatrix}$ , with  $a_{22} \in \mathbb{R}$  and  $a_{22} \neq 0$ .

The Riemannian Lie group  $(G_I, g)$  is locally symmetric for all left invariant metric  $g$  on  $G_I$ .

3.2.2. *Riemannian Lie group  $G_D$ .* If  $D = 0$  then, a left invariant Riemannian metric is equivalent up to automorphism to a metric whose associated matrix is of the form

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{pmatrix} \mu, \nu > 0 \quad \text{or} \quad A_2 = \begin{pmatrix} 1 & \frac{1}{2} & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & \nu \end{pmatrix} \nu > 0.$$

A left invariant metric  $g$  on  $G_0$  is locally symmetric if there exist a Milnor basis relative to  $\langle , \rangle$  which satisfied (9) and (10) with constant structures of the form  $(a, b, c, d) = (a, 0, 0, 0)$ , with  $a > 0$  by Proposition 2.7.

If the matrix of  $\langle , \rangle$  is  $A_1$ , then the above relations yield the following systems:

$$\begin{aligned}
 a a_{12} &= 0 \\
 a a_{22} - (a_{31}a_{12} - a_{11}a_{32} + 2(a_{31}a_{22} - a_{21}a_{32})) &= 0 \\
 a a_{32} &= 0 \\
 a_{31}a_{13} - a_{11}a_{33} + 2(a_{31}a_{23} - a_{21}a_{33}) &= 0 \\
 a_{32}a_{23} - a_{22}a_{33} &= 0 \\
 a_{11}^2 + a_{21}^2\mu + a_{31}^2\nu - 1 &= 0 \\
 a_{11}a_{12} + a_{21}\mu a_{22} + a_{31}\nu a_{32} &= 0 \\
 a_{11}a_{13} + a_{21}\mu a_{23} + a_{31}\nu a_{33} &= 0 \\
 a_{12}^2 + a_{22}^2\mu + a_{32}^2\nu - 1 &= 0 \\
 a_{12}a_{13} + a_{22}\mu a_{23} + a_{32}\nu a_{33} &= 0 \\
 a_{13}^2 + a_{23}^2\mu + a_{33}^2\nu &= 0
 \end{aligned} \tag{14}$$

This system of polynomial equations has no solution. Therefore the metric is not locally symmetric.

If the matrix of  $\langle , \rangle$  is  $A_2$ , then we have the following polynomial system:

$$\begin{aligned}
 aa_{12} &= 0 \\
 aa_{22} - (a_{31}a_{12} - a_{11}a_{32} + 2(a_{31}a_{22} - a_{21}a_{32})) &= 0 \\
 aa_{32} &= 0 \\
 a_{31}a_{13} - a_{11}a_{33} + 2(a_{31}a_{23} - a_{21}a_{33}) &= 0 \\
 a_{32}a_{23} - a_{22}a_{33} &= 0 \\
 \left(a_{11} + \frac{1}{2}a_{21}\right)a_{11} + \left(\frac{1}{2}a_{11} + a_{21}\right)a_{21} + a_{31}^2\nu - 1 &= 0 \\
 \left(a_{11} + \frac{1}{2}a_{21}\right)a_{12} + \left(\frac{1}{2}a_{11} + a_{21}\right)a_{22} + a_{31}\nu a_{32} &= 0 \\
 \left(a_{11} + \frac{1}{2}a_{21}\right)a_{13} + \left(\frac{1}{2}a_{11} + a_{21}\right)a_{23} + a_{31}\nu a_{33} &= 0 \\
 \left(a_{12} + \frac{1}{2}a_{22}\right)a_{12} + \left(\frac{1}{2}a_{12} + a_{22}\right)a_{22} + a_{32}^2\nu - 1 &= 0 \\
 \left(a_{12} + \frac{1}{2}a_{22}\right)a_{13} + \left(\frac{1}{2}a_{12} + a_{22}\right)a_{23} + a_{32}\nu a_{33} &= 0 \\
 \left(a_{13} + \frac{1}{2}a_{23}\right)a_{13} + \left(\frac{1}{2}a_{13} + a_{23}\right)a_{23} + a_{33}^2\nu - 1 &= 0
 \end{aligned} \tag{15}$$

This system of polynomials equations hold for all  $\nu > 0$  and we can choose  $P = \begin{pmatrix} 0 & 0 & \frac{-2}{\sqrt{3}} \\ 0 & 1 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{\nu}} & 0 & 0 \end{pmatrix}$ .

The Riemannian Lie group  $(G_0, g)$  is locally symmetric if and only if the metric  $g$  is equivalent up to automorphism to the metric who associated matrix is  $\begin{pmatrix} 1 & \frac{1}{2} & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & \nu \end{pmatrix}, \nu > 0$ .

If  $D > 1$  then, Any left invariant Riemannian metric on  $G_D$  is equivalent up to automorphism to the metric whose associated matrix is of the form  $\begin{pmatrix} 1 & 1 & 0 \\ 1 & \mu & 0 \\ 0 & 0 & \nu \end{pmatrix}, 1 < \mu \leq D$  and  $\nu > 0$ .

A left invariant metric  $g$  on  $G_D$  is locally symmetric if there exist a Milnor basis relative to  $\langle , \rangle$  which satisfied (9) and (10) with constant structures of the form  $(a, b, c, d) = (a, b, -b, a)$ , with  $a > 0, b > 0$  by Proposition 2.7. Hence, we have the below polynomial

system:

$$\begin{aligned}
a a_{12} + b a_{13} + D(a_{31}a_{22} - a_{21}a_{32}) &= 0 \\
a a_{22} + b a_{23} - (a_{31}a_{12} - a_{11}a_{32} + 2(a_{31}a_{22} - a_{21}a_{32})) &= 0 \\
a a_{32} + b a_{33} &= 0 \\
a a_{13} - b a_{12} + D(a_{31}a_{23} - a_{21}a_{33}) &= 0 \\
a a_{23} - b a_{22} - (a_{31}a_{13} - a_{11}a_{33} + 2(a_{31}a_{23} - a_{21}a_{33})) &= 0 \\
a a_{33} - b a_{32} &= 0 \\
a_{32}a_{23} - a_{22}a_{33} &= 0 \\
a_{32}a_{13} - a_{12}a_{33} + 2(a_{32}a_{23} - a_{22}a_{33}) &= 0 \\
(a_{11} + a_{21})a_{11} + (a_{11} + a_{21}\mu)a_{21} + a_{31}^2\nu - 1 &= 0 \\
(a_{11} + a_{21})a_{12} + (a_{11} + a_{21}\mu)a_{22} + a_{31}\nu a_{32} &= 0 \\
(a_{11} + a_{21})a_{13} + (a_{11} + a_{21}\mu)a_{23} + a_{31}\nu a_{33} &= 0 \\
(a_{12} + a_{22})a_{12} + (a_{12} + a_{22}\mu)a_{22} + a_{32}^2\nu - 1 &= 0 \\
(a_{12} + a_{22})a_{13} + (a_{12} + a_{22}\mu)a_{23} + a_{32}\nu a_{33} &= 0 \\
(a_{13} + a_{23})a_{13} + (a_{13} + a_{23}\mu)a_{23} + a_{33}^2\nu - 1 &= 0
\end{aligned} \tag{16}$$

This system of polynomials equations hold if and only  $\mu = D > 1$  and we can choose

$$P = \begin{pmatrix} 0 & 1 & \frac{-1}{\sqrt{D-1}} \\ 0 & 0 & \frac{1}{\sqrt{D-1}} \\ \frac{1}{\sqrt{\nu}} & 0 & 0 \end{pmatrix}.$$

The Riemannian Lie group  $(G_{D>1}, g)$  is locally symmetric if and only if the metric  $g$  is equivalent up to automorphism to the metric who associated matrix is  $\begin{pmatrix} 1 & 1 & 0 \\ 1 & D & 0 \\ 0 & 0 & \nu \end{pmatrix}, \nu > 0$ .

**3.3. Non simply connected Lie groups.**  $SO(3)$  and  $E_0(2)$  are the non isomorphic non simply connected Lie groups.

Let  $\pi : SU(2) \longrightarrow SO(3)$  and  $p : \widetilde{E}_0(2) \longrightarrow E_0(2)$  be the covering maps. Let  $g$  be a locally symmetric left invariant metric on  $SO(3)$  or  $E_0(2)$ , then  $\pi^*g$  or  $p^*g$  are locally symmetric left invariant metrics. Thus:

**Proposition 3.2.** *We have the following:*

- (1) *Any locally symmetric left invariant Riemannian metric on  $SO(3)$  has the form  $\lambda I_3$ ,  $\lambda > 0$ .*
- (2) *Any locally symmetric left invariant Riemannian metric on  $E_0(2)$  has the form  $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \nu \end{pmatrix}$ ,  $\nu > 0$ .*

**Remark 3.3.** *Any locally symmetric left Riemannian metric  $g$  on  $SO(3)$  is bi-invariant, therefore  $(SO(3), g)$  is a symmetric space.*

#### 4. THE LIE GROUP $E_0(2)$

The Lie Groups  $\mathbb{R}^3$ ,  $\widetilde{E}_0(2)$ ,  $SU(2)$ ,  $G_I$ ,  $G_0$ ,  $G_{D>1}$  are simply connected and admitted locally symmetric left invariant Riemannian metrics, thus are symmetric spaces.  $SO(3)$  is also a symmetric space. In this section we built a family of locally symmetric left invariant Riemannian metrics on  $E_0(2)$  for which  $E_0(2)$  is not a symmetric space.

Recall that  $(\widetilde{E}_0(2), g)$  is a symmetric Riemannian Lie group when the metric  $g$  is equivalent up to automorphism to the metric whose associated matrix is of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \nu \end{pmatrix}, \quad \nu > 0 \text{ with respect to a basis } (\widetilde{X}_1, \widetilde{X}_2, \widetilde{X}_3) \text{ (see [3])}$$

**Lemma 4.1.** *Let  $\gamma : t \mapsto \gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t))$  be the maximal geodesic on  $\widetilde{E}_0(2)$  such that*

$$\gamma(0) = e = \left( \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, 0 \right) \text{ and } \dot{\gamma}(0) = v_1 e_1 + v_2 e_2 + v_3 e_3.$$

Case 1 If  $v_3 = 0$ , then  $\gamma(t) = (v_1 t, v_2 t, 0)$ .

Case 2 If  $v_3 \neq 0$ , then

$$\begin{aligned} & \bullet \gamma(t) = \left( \begin{bmatrix} v_1 t \\ v_2 t \end{bmatrix}, v_3 t \right), \text{ if } \nu = 1, \\ & \bullet \\ \gamma_1(t) &= \frac{v_1}{\left(1 - \frac{1}{\sqrt{\nu}}\right)v_3} \sin\left(1 - \frac{1}{\sqrt{\nu}}\right)v_3 t + \frac{v_2}{\left(1 - \frac{1}{\sqrt{\nu}}\right)v_3} \cos\left(1 - \frac{1}{\sqrt{\nu}}\right)v_3 t - \frac{v_2}{\left(1 - \frac{1}{\sqrt{\nu}}\right)v_3} v_3 t \\ \gamma_2(t) &= \frac{-v_1}{\left(1 - \frac{1}{\sqrt{\nu}}\right)v_3} \cos\left(1 - \frac{1}{\sqrt{\nu}}\right)v_3 t + \frac{v_2}{\left(1 - \frac{1}{\sqrt{\nu}}\right)v_3} \sin\left(1 - \frac{1}{\sqrt{\nu}}\right)v_3 t + \frac{v_1}{\left(1 - \frac{1}{\sqrt{\nu}}\right)v_3} v_3 t \\ \gamma_3(t) &= v_3 t \end{aligned} \quad \text{if } \nu \neq 1. \quad (17)$$

**proof:** Let  $(e_1, e_2, e_3)$  be the orthonormal basis such that  $e_1 = X_1$ ,  $e_2 = X_2$ ,  $e_3 = \frac{1}{\sqrt{\nu}} X_3$ .

The Lie bracket are defined by:

$$[e_1, e_2] = 0, \quad [e_2, e_3] = \frac{-1}{\sqrt{\nu}} e_1, \quad [e_3, e_1] = \frac{-1}{\sqrt{\nu}} e_2.$$

$g$  is not a bi-invariant metric. If  $\alpha : t \mapsto \alpha(t)$  is the associated curve in  $\mathfrak{g}$ , by (6)  $\dot{\gamma}(0) = L_{\gamma(0)*} \alpha(0) = \alpha(0)$ , hence we have  $\alpha(0) = v_1 e_1 + v_2 e_2 + v_3 e_3$ . Setting  $\alpha(t) = \alpha_1(t) e_1 + \alpha_2(t) e_2 + \alpha_3(t) e_3$ ,  $\dot{\alpha}(t) = \alpha'_1(t) e_1 + \alpha'_2(t) e_2 + \alpha'_3(t) e_3$ . The equation (8) is equivalent to the following system:

$$\begin{aligned} \alpha'_1(t) &= \frac{-1}{\sqrt{\nu}} \alpha_2(t) \alpha_3(t) \\ \alpha'_2(t) &= \frac{1}{\sqrt{\nu}} \alpha_1(t) \alpha_3(t) \\ \alpha'_3(t) &= 0 \end{aligned} \quad (18)$$

with the initial condition  $\alpha(0) = v_1 e_1 + v_2 e_2 + v_3 e_3$ . Therefore:

$$\bullet \alpha(t) = (v_1, v_2, 0) \quad \text{if } v_3 = 0 \quad \text{and}$$

$\bullet$

$$\begin{aligned} \alpha_1(t) &= v_1 \cos\left(\frac{v_3}{\sqrt{\nu}} t\right) - v_2 \sin\left(\frac{v_3}{\sqrt{\nu}} t\right) \\ \alpha_2(t) &= v_2 \cos\left(\frac{v_3}{\sqrt{\nu}} t\right) + v_1 \sin\left(\frac{v_3}{\sqrt{\nu}} t\right) \\ \alpha_3(t) &= v_3 \end{aligned} \quad \text{if } v_3 \neq 0. \quad (19)$$

The product in  $\widetilde{E}_0(2)$  is defined by:

$$\left( \begin{bmatrix} x \\ y \end{bmatrix}, s \right) \cdot \left( \begin{bmatrix} x' \\ y' \end{bmatrix}, s' \right) = \left( \begin{bmatrix} x \\ y \end{bmatrix} + R(s) \begin{bmatrix} x' \\ y' \end{bmatrix}, s + s' \right)$$

where  $R(s) = \begin{pmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{pmatrix}$ . Let  $\gamma(t) = \left( \begin{bmatrix} \gamma_1(t) \\ \gamma_2(t) \end{bmatrix}, \gamma_3(t) \right)$ , the differential of  $L_{\gamma(t)}$  at the identity element  $e$ , is given by

$$[L_{\gamma(t)} * e] = \begin{pmatrix} \cos(\gamma_3(t)) & \sin(\gamma_3(t)) & 0 \\ -\sin(\gamma_3(t)) & \cos(\gamma_3(t)) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and the equation (6)  $\dot{\gamma}(t) = L_{\gamma(t)*e}\alpha(t)$  is equivalent to the system:

$$\begin{aligned} \gamma'_1(t) &= \alpha_1(t) \cos(\gamma_3(t)) + \alpha_2(t) \sin(\gamma_3(t)) \\ \gamma'_2(t) &= -\alpha_1(t) \sin(\gamma_3(t)) + \alpha_2(t) \cos(\gamma_3(t)) \\ \gamma'_3(t) &= \alpha_3(t) = v_3 \end{aligned} \quad (20)$$

Then the result follow by solving the system (20). ■

**Lemma 4.2.** *The local geodesic symmetry  $S_e$  at identity  $e \in \widetilde{E}_0(2)$  is defined by:*

$$S_e \left( \left( \begin{bmatrix} x \\ y \end{bmatrix}, s \right) \right) = \left( \begin{bmatrix} -x \cos \left( 1 - \frac{1}{\sqrt{v}} \right) s - y \sin \left( 1 - \frac{1}{\sqrt{v}} \right) s \\ x \sin \left( 1 - \frac{1}{\sqrt{v}} \right) s - y \cos \left( 1 - \frac{1}{\sqrt{v}} \right) s \end{bmatrix}, -s \right). \quad (21)$$

**proof:** We then define, for  $v = v_1 e_1 + v_2 e_2 + v_3 e_3 \in \mathfrak{g}$ , the exponential map at  $e$ :

$$\text{If } v = 1, \quad \exp_e(v) = \left( \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, v_3 \right)$$

If not ( $v \neq 1$ ),

$$\exp_e(v) = \begin{cases} \left( \begin{bmatrix} \frac{v_1}{\left(1 - \frac{1}{\sqrt{v}}\right)v_3} \sin\left(1 - \frac{1}{\sqrt{v}}\right)v_3 + \frac{v_2}{\left(1 - \frac{1}{\sqrt{v}}\right)v_3} \cos\left(1 - \frac{1}{\sqrt{v}}\right)v_3 - \frac{v_2}{\left(1 - \frac{1}{\sqrt{v}}\right)v_3} \\ \frac{-v_1}{\left(1 - \frac{1}{\sqrt{v}}\right)v_3} \cos\left(1 - \frac{1}{\sqrt{v}}\right)v_3 + \frac{v_2}{\left(1 - \frac{1}{\sqrt{v}}\right)v_3} \sin\left(1 - \frac{1}{\sqrt{v}}\right)v_3 + \frac{v_1}{\left(1 - \frac{1}{\sqrt{v}}\right)v_3} \end{bmatrix}, v_3 \right) & \text{if } v_3 \neq 0 \\ \left( \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, 0 \right) & \text{if } v_3 = 0 \end{cases} \quad (22)$$

The inverse map of  $\exp_e$  then follow.

For  $v = 1$ , the inverse map  $(\exp_e)^{-1}$  of  $\exp_e$  is defined by

$$(\exp_e)^{-1} \left( \left( \begin{bmatrix} x \\ y \end{bmatrix}, s \right) \right) = (x e_1 + y e_2 + s e_3).$$

and the local geodesic symmetry at  $e$  is given by:

$$S_e = \exp_e \circ (-Id) \circ (\exp_e)^{-1} \left( \left( \begin{bmatrix} x \\ y \end{bmatrix}, s \right) \right) = \left( \begin{bmatrix} -x \\ -y \end{bmatrix}, -s \right) \quad (23)$$

For  $v \neq 1$

$$V = \left\{ v_1 e_1 + v_2 e_2 + v_3 e_3 \in \mathfrak{g}, v_1, v_2 \in \mathbb{R}, v_3 \in \left[ \frac{-2\pi}{1 - \frac{1}{\sqrt{v}}}, \frac{2\pi}{1 - \frac{1}{\sqrt{v}}} \right] \right\}$$

is and open set in  $\mathfrak{g}$ . The image  $U$  of  $V$  by  $\exp_e$  is an open set. In fact, the Riemannian Lie group  $\widetilde{E}_0(2)$  endowed with the above metric is complete and the sectional curvature  $K = 0$  since the metric is locally symmetric.  $K$  non positive; by Cartan-Hadamard

theorem,  $\exp_e$  is a covering map, (see [1]) therefore  $\exp_e$  is an open map . Hence,  $\exp_e : V \longrightarrow U$  is one-to-one .  $(\exp_e)^{-1}$  is defined for  $\left(\begin{bmatrix} x \\ y \end{bmatrix}, s\right) \in U$  by:

$$(\exp_e)^{-1}\left(\left(\begin{bmatrix} x \\ y \end{bmatrix}, s\right)\right) = \begin{cases} \frac{\left(1 - \frac{1}{\sqrt{v}}\right)s}{2\left(1 - \cos\left(1 - \frac{1}{\sqrt{v}}\right)s\right)} \left(x \sin\left(1 - \frac{1}{\sqrt{v}}\right)s - y \cos\left(1 - \frac{1}{\sqrt{v}}\right)s + y\right) e_1 + \\ \frac{\left(1 - \frac{1}{\sqrt{v}}\right)s}{2\left(1 - \cos\left(1 - \frac{1}{\sqrt{v}}\right)s\right)} \left(x \cos\left(1 - \frac{1}{\sqrt{v}}\right)s + y \sin\left(1 - \frac{1}{\sqrt{v}}\right)s - x\right) e_2 + s e_3 & \text{if } s \neq 0 \\ x e_1 + y e_2 & \text{if } s = 0 \end{cases} .$$

The local geodesic symmetry is given by:

$$\begin{aligned} S_e\left(\left(\begin{bmatrix} x \\ y \end{bmatrix}, s\right)\right) &= \exp_e \circ (-Id) \circ (\exp_e)^{-1}\left(\left(\begin{bmatrix} x \\ y \end{bmatrix}, s\right)\right) \\ &= \left(\begin{bmatrix} -x \cos\left(1 - \frac{1}{\sqrt{v}}\right)s - y \sin\left(1 - \frac{1}{\sqrt{v}}\right)s \\ x \sin\left(1 - \frac{1}{\sqrt{v}}\right)s - y \cos\left(1 - \frac{1}{\sqrt{v}}\right)s \end{bmatrix}, -s\right). \end{aligned} \quad (24)$$

■

**Remark 4.3.** (1) It is obvious that the local geodesic symmetry  $S_e$  is defined on the hold  $\widetilde{E}_0(2)$ , it is a global involution and it has a single fix point.

(2) Since  $\widetilde{E}_0(2)$  is a symmetric space, we can now compute the expression of the geodesic symmetry for all  $\widetilde{x} = \left(\begin{bmatrix} a \\ b \end{bmatrix}, c\right)$ . The inverse  $\widetilde{x}^{-1}$  of  $\widetilde{x}$  is

$$\widetilde{x}^{-1} = \left(R(-c) \begin{bmatrix} -a \\ -b \end{bmatrix}, -c\right).$$

and

$$S_{\widetilde{x}}\left(\begin{bmatrix} x \\ y \end{bmatrix}, s\right) = \left(\begin{bmatrix} a \\ b \end{bmatrix} + R\left(\left(1 - \frac{1}{\sqrt{v}}\right)(s - c)\right) \begin{bmatrix} a - x \\ b - y \end{bmatrix}, 2c - s\right)$$

For the convenience, in the rest of this paper, elements of the universal covering are denoted  $\widetilde{x}$ . The covering map  $p : \widetilde{E}_0(2) \longrightarrow E_0(2)$  is defined by:

$$p\left(\begin{bmatrix} x \\ y \end{bmatrix}, s\right) = \left(\begin{bmatrix} x \\ y \end{bmatrix}, R(s)\right) \quad \text{where} \quad R(s) = \begin{pmatrix} \cos(s) & \sin(s) \\ -\sin(s) & \cos(s) \end{pmatrix},$$

(see [4]).  $\widetilde{U} = \mathbb{R}^2 \times ]-\pi, \pi[$  is an open neighborhood of the identity  $\widetilde{e} = \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, 0\right)$  in  $\widetilde{E}_0(2)$ , its image by the covering map  $p$ , denoted  $U = \mathbb{R}^2 \times \{R(s), s \in ]-\pi, \pi[\}$ , is an open neighborhood of  $e$  in  $E_0(2)$ . Setting  $p_1 = p|_{\widetilde{U}}$ ,  $p_1$  is one-to-one from  $\widetilde{U}$  to  $U$ .  $dp(\widetilde{e}) = I_3$  is an automorphism of  $\mathfrak{g}$ .

**Lemma 4.4.** The local geodesic symmetry  $S_e$  at  $e \in E_0(2)$  is defined by:

$$S_e\left(\left(\begin{bmatrix} x \\ y \end{bmatrix}, R(s)\right)\right) = \left(\begin{bmatrix} -x \cos\left(1 - \frac{1}{\sqrt{v}}\right)s - y \sin\left(1 - \frac{1}{\sqrt{v}}\right)s \\ x \sin\left(1 - \frac{1}{\sqrt{v}}\right)s - y \cos\left(1 - \frac{1}{\sqrt{v}}\right)s \end{bmatrix}, R(-s)\right). \quad (25)$$

**proof:** Equality (25) hold by direct computation of  $S_e = p_1 \circ S_{\bar{e}} \circ p_1^{-1}$  for  $\left(\begin{bmatrix} x \\ y \end{bmatrix}, R(s)\right) \in U$  ■

**4.1. Proof of Theorem 1.3. proof:** If  $S_e$  is a global symmetry on  $E_0(2)$ , then the lift  $\widetilde{S}_e$  of  $S_e$  is a global isometry on  $\widetilde{E}_0(2)$  ( see[6]), and we have the following commutative diagram

$$\begin{array}{ccc} \widetilde{E}_0(2) & \xrightarrow{\widetilde{S}_e} & \widetilde{E}_0(2) \\ p \downarrow & & \downarrow p \\ E_0(2) & \xrightarrow{S_e} & E_0(2) \end{array}$$

If  $\widetilde{S}_e(\bar{e}) = \bar{e}_1$  with  $\bar{e} \neq \bar{e}_1$ , then  $dS_e(\bar{e}) = -Id : T_{\bar{e}}\widetilde{E}_0(2) \rightarrow T_{\bar{e}_1}\widetilde{E}_0(2)$  with  $T_{\bar{e}}\widetilde{E}_0(2) \cap T_{\bar{e}_1}\widetilde{E}_0(2) = \emptyset$ , it follow that,  $\widetilde{S}_e(\bar{e}) = \bar{e}$ . Since any isometry is determined by its value and its derivative at a single point,  $\widetilde{S}_e = S_{\bar{e}}$

If  $\left(\begin{bmatrix} x \\ y \end{bmatrix}, R(s)\right) \in E_0(2)$ , then  $s \in ]-\pi, \pi]$  and

$$p^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}, R(s)\right) = \left\{\left(\begin{bmatrix} x \\ y \end{bmatrix}, s + 2k\pi\right), k \in \mathbb{Z}\right\}.$$

Therefore the image of  $\left(\begin{bmatrix} x \\ y \end{bmatrix}, R(s)\right)$  by  $S_e$  follow:

$$S_e\left(\left(\begin{bmatrix} x \\ y \end{bmatrix}, R(s)\right)\right) = \left(\begin{bmatrix} -x \cos\left(1 - \frac{1}{\sqrt{v}}\right)(s + 2k\pi) - y \sin\left(1 - \frac{1}{\sqrt{v}}\right)(s + 2k\pi) \\ x \sin\left(1 - \frac{1}{\sqrt{v}}\right)(s + 2k\pi) - y \cos\left(1 - \frac{1}{\sqrt{v}}\right)(s + 2k\pi) \end{bmatrix}, R(-s)\right). \quad (26)$$

$S_e$  is defined on the hold  $E_0(2)$  and unique if and only if

$$\begin{aligned} \left(1 - \frac{1}{\sqrt{v}}\right)(s + 2k\pi) &= \left(1 - \frac{1}{\sqrt{v}}\right)s + 2k'\pi. &\iff & \left(1 - \frac{1}{\sqrt{v}}\right)2k\pi = 2k'\pi \\ & &\iff & 1 - \frac{1}{\sqrt{v}} \in \mathbb{Z} \\ & &\iff & -\frac{1}{\sqrt{v}} \in \mathbb{Z} \\ & &\iff & \frac{1}{\sqrt{v}} \in \mathbb{N} \setminus \{0\} \quad \text{since } v > 0. \end{aligned} \quad (27)$$

This completes the proof. ■

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